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Necessary and Sufficient Condition for Oscillation of a Neutral Differential System with Several Delays

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In this paper we prove that a general neutral differential system with several delays has a nonoscillatory solution if and only if its characteristic equations has a real root. © 1989 Academic Press, Inc.

1. INTRODUCTION

Recently there have been several papers concerning the study of delay differential equations whose solutions exhibit oscillatory behavior. The main reason for this interest is that delay differential equations play an important role in applications, and—for instance—in biological applications delay equations give a better description of fluctuations in population than the ordinary ones (see, e.g., [3, 5, 8] and references cited therein).

In the oscillation theory of linear delay differential equations one of the most important objectives is to give a necessary and sufficient condition for oscillation via the characteristic equation. Such a result for scalar delay differential equations was proved in [1, 7 and 9] using various methods,

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and the method of [1] was extended to delay systems in [2]. In the neutral case many special scalar equations have been investigated using different techniques (see, e.g., [3, 6] and references therein), but at this time we do not know about a general result.

In this paper we prove that in the general scalar as well as system cases a neutral delay differential equation with several delays has a non-oscillatory solution if and only if its characteristic equation has a real root. Our proof is elementary and is based on the method of the Laplace transform, using a fact from [4] that the solutions of a neutral equation are not growing faster than exponentially.

2. The Main Result and Some Corollaries

Consider the neutral delay differential equation in the form

$$\frac{d}{dt}\left[x(t)-\sum_{j=1}^{m} B_j x(t-\sigma_j)\right]=\sum_{i=1}^{k} A_i x(t-\tau_i), \tag{2.1}$$

where we assume

(A) A_i, $(1 \leqslant i \leqslant k)$ and B_j, $(1 \leqslant j \leqslant m)$ are given n by n matrices, τ_i, $(1 \leqslant i \leqslant k)$ and σ_j, $(1 \leqslant j \leqslant m)$ are given positive constants, $\gamma=\max\{\tau, \sigma\}$, $\tau=\max_{1 \leqslant i \leqslant k}\tau_i$, $\sigma=\max_{1 \leqslant j \leqslant m}\sigma_j$.

Definition 2.1. By a solution to (2.1) on the interval $[-\gamma, \infty)$, we mean a function $x \in C([-\gamma, \infty), R^n)$ such that $x(t)-\sum_{j=1}^{m} B_j x(t-\sigma_j)$ is continuously differentiable and satisfies (2.1) on $[0, \infty)$.

Definition 2.2. A solution $x=(x_1, ..., x_n)^T\colon [-\gamma, \infty) \to R^n$ of (2.1) is called nonoscillatory if there exist $t_0 \geqslant 0$ and $i_0 \in \{1, ..., n\}$ such that $|x_{i_0}(t)|>0$, $t \geqslant t_0$.

Remark 2.1. If Eq. (2.1) has a nonoscillatory solution then there is a solution $x=(x_1, ..., x_n)^T\colon [-\gamma, \infty) \to R^n$ of (2.1) such that for some $i_0 \in \{1, ..., n\}$, $x_{i_0}(t)>0$, $t \geqslant -\gamma$. In fact, if $y(t)$ is a nonoscillatory solution of (2.1) then there is an index $i_0 \in \{1, ..., n\}$ and there is a $t_0 \geqslant 0$ such that $|y_{i_0}(t)|>0$, $t \geqslant t_0$. But it is easily seen that $x(t)=(\text{sign } y_{i_0}(t_0))\, y(t+\gamma+t_0)$ is a solution of (2.1) and $x_{i_0}(t)=|y_{i_0}(t)|>0$, $t \geqslant -\gamma$.

We are now prepared to state and prove our main result.

Theorem 2.1. *Assume* (A) *holds. Then Eq.* (2.1) *has a nonoscillatory solution if and only if its characteristic equation*

$$\det\left(\lambda I-\lambda\sum_{j=1}^{m}B_je^{-\lambda\sigma_j}-\sum_{i=1}^{n}A_je^{-\lambda\tau_i}\right)=0 \tag{2.2}$$

has a real root.

Proof. If Eq. (2.2) has a real root λ then there exists a constant vector $c=(c_1, ..., c_n)^T$ such that $\sum_{i=1}^{n}|c_i|>0$ and

$$x(t)=e^{\lambda t}c \tag{2.3}$$

is a solution of (2.1) on $(-\infty, \infty)$. But it is easily seen that $x(t)$ is a nonoscillatory solution of (2.1).

Therefore the proof will be complete if we show that if Eq. (2.1) has a nonoscillatory solution on $[-\gamma, \infty)$ then Eq. (2.2) has a real root. To this end, we assume the opposite case, that is, (2.1) has a nonoscillatory solution on $[-\gamma, \infty)$ and parallel to this (2.2) has no real root. We show that it leads to a contradiction.

By Remark 2.1, we have that if (2.1) has a nonoscillatory solution then it has a solution $x(t)=(x_1(t), ..., x_n(t))^T$ such that for an index $i_0\in\{1, ..., n\}$,

$$x_{i_0}(t)>0, \qquad t\geqslant -\gamma. \tag{2.4}$$

But from [4], we know that the solutions of (2.1) are not growing faster than exponentials, therefore

$$|x_i(t)|\leqslant ae^{bt}, \qquad t\geqslant -\gamma, \quad 1\leqslant i\leqslant n, \tag{2.5}$$

for some constants $a>0$ and $b\in R$.

Thus the Laplace transform $\mathscr{L}[x]$ defined by

$$\mathscr{L}[x](s)=\int_0^\infty e^{-st}x(t)\,dt \tag{2.6}$$

exists and is an analytic function of s for Re $s>b$. But (2.5) implies that the function

$$u(t)=x(t)-\sum_{j=1}^{m}B_jx(t-\sigma_j), \qquad t\geqslant 0,$$

and its derivative $\dot{u}(t)$ have the same exponential growth b as $x(t)$. Thus

$$\mathscr{L}[\dot{u}](s)=s\mathscr{L}[u](s)-u(0)$$

exists for Re $s > b$, and from (2.1) it follows that

$$s\mathscr{L}[x](s) - s\sum_{j=1}^{m} B_j e^{-s\sigma_j}\mathscr{L}[x](s) - x(0) + \sum_{j=1}^{m} B_j x(-\sigma_j)$$
$$- s\sum_{j=1}^{m} B_j e^{-s\sigma_j}\int_{-\sigma_j}^{0} x(t)\,e^{-st}\,dt$$
$$= \sum_{i=1}^{k} A_i e^{-s\tau_i}\mathscr{L}[x](s) + \sum_{i=1}^{k} A_i e^{-s\tau_i}\int_{-\tau_i}^{0} x(t)\,e^{-st}\,dt, \qquad \text{Re } s > b. \quad (2.7)$$

Here, we used the simple fact

$$\mathscr{L}[x(\cdot - \kappa)](s) = \int_0^\infty e^{-st}x(t-\kappa)\,dt$$
$$= e^{-s\kappa}\mathscr{L}[x](s) + e^{-s\kappa}\int_{-\kappa}^{0} x(t)\,e^{-st}\,dt, \qquad \text{Re } s > b,$$

for any constant $\kappa \in [0, \gamma]$.

From (2.7), we have

$$\Delta(s)\,\mathscr{L}[x](s) = \varphi(s), \qquad \text{Re } s > b, \tag{2.8}$$

where

$$\Delta(s) = sI - s\sum_{j=1}^{m} B_j e^{-s\sigma_j} - \sum_{i=1}^{k} A_i e^{-s\tau_i} \tag{2.9}$$

and

$$\varphi(s) = x(0) - \sum_{j=1}^{m} B_j x(-\sigma_j) + s\sum_{j=1}^{m} B_j e^{-s\sigma_j}\int_{-\sigma_j}^{0} x(u)\,e^{-su}\,du$$
$$+ \sum_{i=1}^{k} A_i e^{-s\tau_i}\int_{-\tau_i}^{0} x(u)\,e^{-su}\,du \tag{2.10}$$

are analytic functions of s for all s.

Since (2.2) has no real root, we have

$$a(s) = \det \Delta(s) \neq 0, \qquad s \in R,$$

therefore (2.8) implies

$$\mathscr{L}[x](s) = \Delta^{-1}(s)\,\varphi(s) \tag{2.11}$$

for any $s \in [b, \infty)$, where the right-hand side is analytic on R. Denote by $\psi_{i_0}(s)$ the i_0th component of $\Delta^{-1}(s)\,\varphi(s)$, that is $\psi_{i_0}(s) = (\Delta^{-1}(s)\,\varphi(s))_{i_0}$.

Then (2.11) yields

$$\mathscr{L}[x_{i_0}](s)=\psi_{i_0}(s) \tag{2.12}$$

for Re $s>b$.

Formula (2.12) shows that $\mathscr{L}[x_{i_0}]$ has an analytic continuation on the whole real axis. What we will show now is that this continuation can in fact be expressed as a Laplace transform, that is, for every s in $\mathbb{R}$, $e^{-st}x_{i_0}(t)$ is in $\mathscr{L}^1(\mathbb{R}^+)$.

We may first observe that the set

$$\{s\in\mathbb{R}: e^{-st}x_{i_0}(t)\in\mathscr{L}^1(\mathbb{R}^+)\}$$

is an interval $(s_0, +\infty)$.

On this interval, we have

$$\int_0^\infty e^{-st}x_{i_0}(t)\,dt=\psi_{i_0}(s),$$

and because ψ_{i_0} is continuous and $x_{i_0}\geqslant 0$, it follows from Beppo-Levi's theorem that the integral is also finite for $s=s_0$ (if $s_0>-\infty$). Let us assume for the moment that $s_0>-\infty$. We choose a point $s_1>s_0$, such that the Taylor series associated to ψ_{i_0} at $s=s_1$ has a radius of convergence which exceeds s_1-s_0. We can express the coefficients of the Taylor series of ψ_{i_0} at s_1 in terms of x_{i_0}:

$$\frac{\psi_{i_0}^{(k)}(s_1)}{k!}=\frac{(-1)^k}{k!}\int_0^\infty t^k\cdot e^{-s_1t}\cdot x_{i_0}(t)\,dt. \tag{2.13}$$

Inside the disk of convergence, the series

$$\sum_{k\geqslant 0}\frac{|\psi_{i_0}^{(k)}(s_1)|}{k!}(s_1-s)^k$$

also converges. This implies that

$$\sum_{k\geqslant 0}\frac{1}{k!}(s_1-s)^k\int_0^\infty t^k\cdot e^{s_1t}\cdot x_{i_0}(t)\,dt<+\infty \tag{2.14}$$

for s in some interval $(s_1-\delta, s_0)$, for some $\delta>0$.

But this expression can be seen as a sum of integrals of positive functions. Beppo-Levi's theorem applies then and leads to the conclusion that

$$\int_0^\infty e^{-st}x_{i_0}(t)\,dt<+\infty$$

for s in $(s_0-\delta, s_0)$.

This is a contradiction with the definition s_0, and leads to the conclusion that Formula (2.12) holds on the whole real axis.

Now consider the behavior of $\psi_{i_0}(s)$ as $s \to -\infty$. From (2.10), it is easily seen that there exist $a_1 > 0$ and $a_2 > 0$ such that

$$|\varphi(s)| \leqslant a_1 e^{a_2|s|}, \qquad s \in R. \tag{2.15}$$

One can see that for any n by n matrix A, $\det A > 0$ implies

$$|A^{-1}| \leqslant \frac{b}{\det A} |A|^{n-1},$$

where b is a constant depending only on n. Thus

$$|\Delta^{-1}(s)| \leqslant \frac{b}{a(s)} |\Delta(s)|^{n-1}, \qquad s \in R, \tag{2.16}$$

where $a(s) = \det \Delta(s) > 0$. But (2.9) implies that for some constants $c_1 > 0$ and $c_2 > 0$,

$$|\Delta(s)| \leqslant c_1 e^{c_2|s|}, \qquad s \in R. \tag{2.17}$$

Moreover, $a(s)$ is a polynomial of s, $e^{-s\sigma_j}$, $(1 \leqslant j \leqslant m)$, and $e^{-s\tau_i}$, $(1 \leqslant i \leqslant k)$, where $\sigma_j > 0$, $(1 \leqslant j \leqslant m)$, and $\tau_i > 0$, $(1 \leqslant i \leqslant k)$. On the other hand $a(s) > 0$, $s \in R$, therefore $\lim_{s \to -\infty} a(s) = +\infty$. Thus there exists $s_1 \in (-\infty, 0)$ such that

$$\frac{1}{a(s)} \leqslant 1, \qquad s \leqslant s_1.$$

Thus (2.16) and (2.17) imply

$$|\Delta^{-1}(s)| \leqslant b(c_1)^{n-1} e^{(n-1)c_2|s|}, \qquad s \leqslant s_1,$$

and (2.15) yields

$$|\Delta^{-1}(s)\, \varphi(s)| \leqslant d_1 e^{d_2|s|}, \qquad s \leqslant s_1,$$

where $d_1 > 0$ and $d_2 > 0$ are suitable constants. Since (2.12) holds for all s, it follows that

$$0 < \mathscr{L}[x_{i_0}](s) = \psi_{i_0}(s) \leqslant |\Delta^{-1}(s)\, \varphi(s)| \leqslant d_1 e^{d_2|s|},$$

for any $s \leqslant s_1$, therefore

$$0 \leqslant \lim_{s \to -\infty} \sup e^{-d_2|s|} \mathscr{L}[x_{i_0}](s) \leqslant d_1. \tag{2.18}$$

But for $T > d_2$, we obtain

$$\begin{aligned} e^{-d_2|s|}\mathscr{L}[x_{i_0}](s) &= e^{-d_2|s|}\int_0^\infty e^{-st}x_{i_0}(t)\,dt \\ &\geqslant e^{-d_2|s|}\int_T^\infty e^{-st}x_{i_0}(t)\,dt \\ &\geqslant e^{-d_2|s|}e^{T|s|}\int_T^\infty x_{i_0}(t)\,dt \to +\infty \end{aligned}$$

as $s \to -\infty$, since $x_{i_0}(t) > 0$ $(t \geqslant 0)$. This contradicts (2.18), therefore Eq. (2.2) indeed has a real root. The proof of the theorem is complete.

In the scalar case $(n=1)$, we have

COROLLARY 2.1. *Assume b_j and $\sigma_j \geqslant 0$ $(1 \leqslant j \leqslant m)$ and a_i and $\tau_i \geqslant 0$ $(1 \leqslant i \leqslant k)$ are given real numbers. Then the equation*

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{m} b_j x(t-\sigma_j)\right] = \sum_{i=1}^{k} a_i x(t-\tau_i)$$

is oscillatory if and only if its characteristic equation

$$\lambda - \lambda \sum_{j=1}^{m} b_j e^{-\lambda\sigma_j} = \sum_{i=1}^{k} a_i e^{-\lambda\tau_i}$$

has no real root.

Since any higher order equation is equivalent to a first order system, we have

COROLLARY 2.2. *Let $n_k \geqslant 0$ $(0 \leqslant k \leqslant n)$ be given integers and assume a_k, $p_{k,j}$ $(0 \leqslant k \leqslant n,\ 0 \leqslant j \leqslant n_k)$ and $\tau_{k,j} \geqslant 0$ $(1 \leqslant k \leqslant n,\ 0 \leqslant j \leqslant n_k)$ are given real numbers. Then the equation*

$$\sum_{k=1}^{n} \frac{d^k}{dt^k}\left[a_k x(t) - \sum_{j=1}^{n_k} p_{k,j} x(t-\tau_{k,j})\right] = \sum_{j=1}^{n_0} p_{0,j} x(t-\tau_{0,j})$$

has a nonoscillatory solution if and only if its characteristic equation

$$\sum_{k=1}^{n} \lambda^k \left[a_k - \sum_{j=1}^{n_k} p_{k,j} e^{-\lambda\tau_{k,j}}\right] = \sum_{j=1}^{n_0} p_{0,j} e^{-\lambda\tau_{0,j}}$$

has a real root.

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